On a characteristic initial value problem in plasma physics

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Abstract

The relativistic Vlasov-Maxwell system of plasma physics is considered with initial data on a past light cone. This characteristic initial value problem arises in a natural way as a mathematical framework to study the existence of solutions isolated from incoming radiation. Various consequences of the mass-energy conservation and of the absence of incoming radiation condition are first derived assuming the existence of global smooth solutions. In the spherically symmetric case, the existence of a unique classical solution in the future of the initial cone follows by arguments similar to the case of initial data at time $t=0$. The total mass-energy of spherically symmetric solutions equals the (properly defined) mass-energy on backward and forward light cones.

1 Introduction

In a system of Cartesian coordinates (t,x), $t\in\mathbb{R}, x\in\mathbb{R}^3$, the Vlasov-Maxwell system is given by

$$\partial_t f+\widehat{p}\cdot\nabla_x f+(E+\widehat{p}\times B)\cdot\nabla_p f=0, \tag{1.1}$$

$$\partial_t E-\nabla\times B=-j, \quad \partial_t B+\nabla\times E=0, \tag{1.2}$$

$$\nabla\cdot E=\rho, \quad \nabla\cdot B=0, \tag{1.3}$$

$$\rho(t,x)=\int f(t,x,p)\,dp, \quad j(t,x)=\int \widehat{p} f(t,x,p)\,dp. \tag{1.4}$$

The Vlasov-Maxwell system models the dynamics of collisionless plasmas. We consider for simplicity a plasma consisting of a single species of particle. The unknowns are the particle density in phase-space, $f=f(t,x,p)$, where $p\in\mathbb{R}^3$ is the momentum variable, and the mean electromagnetic field $(E,B)=(E,B)(t,x)$ generated by the particles. The expression

$$\widehat{p}=\frac{p}{\sqrt{1+|p|^2}}$$

denotes the relativistic velocity of a particle with momentum p. Units are chosen such that the mass and the charge of each particle and the speed of light are equal to unity. The simbol $\times$ denotes the usual vector product in $\mathbb{R}^3$. We refer to [3, 5, 9, 11, 12, 14, 15, 16, 17] for background on the Cauchy problem for the Vlasov-Maxwell system. Classical solutions of (1.1)–(1.4) satisfy the energy identity

$$\partial_t e + \nabla \cdot \mathfrak{p} = 0, \tag{1.5}$$

where

$$e(t,x) = \int \sqrt{1+|p|^2}\, f\, dp + \frac{1}{2}|E|^2 + \frac{1}{2}|B|^2, \quad \mathfrak{p}(t,x) = \int p\, f\, dp + E \times B.$$

Integrating (1.5) one obtains the conservation of the total energy

$$M(t) = \int\int \sqrt{1+|p|^2} f\, dp\, dx + \frac{1}{2}\int \left(|E|^2 + |B|^2\right) dx = const. \tag{1.6}$$

Solutions of Vlasov-Maxwell also satisfy the continuity equation

$$\partial_t \rho + \nabla \cdot j = 0, \tag{1.7}$$

which upon integration leads to the conservation of the total (rest) mass

$$N(t) = \int\int f\, dp\, dx = const. \tag{1.8}$$

The purpose of this paper is to set up a mathematical framework for the analysis of solutions to the Vlasov-Maxwell system which satisfy the no-incoming radiation condition, that is

$$\lim_{r\to\infty} \int_{v_1}^{v_2} \int_{|x|=r} k \cdot [E \times B](v - |x|, x)\, dS_r(x)\, dv = 0, \tag{1.9}$$

for all $v_1, v_2 \in \mathbb{R}$, where dS_r is the surface element on the sphere of radius r and $k = x/r$ is the unit normal on this sphere. This corresponds to the physical condition that the electromagnetic field carries no energy to the past null infinity of Minkowski space, see [5, 6].

Solutions of Vlasov-Maxwell isolated from incoming radiation were first studied in [5]. The result of [5] is that such solutions exist globally in time for small data of the Cauchy problem (i.e., data at time $t=0$). However the Cauchy problem is not a natural framework to generate solutions isolated from incoming radiation. In fact, since the no-incoming radiation condition is imposed at $t \to -\infty$, there is no meaningful notion of local isolated solution with data at $t=0$. Therefore, in the framework of the Cauchy problem, one can only prove the existence of global (or semiglobal) solutions which satisfy (1.9). This requires the use of uniform in time *a priori* estimates, which are not available in general for non-linear problems.

A more natural setting for the study of isolated solutions is the initial value problem with data on a surface which cuts past null infinity. Examples of

such surfaces are past light cones and backward hyperboloids. This paper is concerned with the first case.

Another motivation for studying the initial value problem with data on a past light cone comes from physical grounds. The initial data correspond to the outcome of an experimental measurement on the state of the physical system at the present time; the existence of a unique solution with the given data assures that the outcome of any future measurement is predicted by the theory. If this physical interpretation of the initial value problem is accepted, then it is clear that the initial data for relativistic models, such as the Vlasov-Maxwell system, should be given on a past light cone. In fact the set of events which are accessible to an observer at the proper time $t=0$ lie on the past light cone with vertex on the world line of the observer at $t=0$. The state of the system on the surface $t=0$, on the other hand, cannot be measured, because these events form a spacelike hypersurface in Minkowski space. Such a discrepancy between the "physical" and the "mathematical" initial value problem has been sometimes discussed in the physical literature, see [7, 8, 10] and the references therein.

In order to study the Vlasov-Maxwell system with initial data on a past light cone, we first rewrite the equations in the coordinates (v,x), where $x\in\mathbb{R}^3$ and $v\in\mathbb{R}$ is the advanced time, which is defined by the condition that the surfaces $v=$constant correspond to the past light cones with vertex on the timelike curve $|x|=0$ (the world-line of the observer). Denote by $f_\wedge=f_\wedge(v,x,p)$, $(E_\wedge,B_\wedge)=(E_\wedge,B_\wedge)(v,x)$ the particle density and the electromagnetic field expressed in these coordinates. They are related to the solutions of (1.1)–(1.4) by $f_\wedge(v,x,p)=f(v-|x|,x,p)$, $(E_\wedge,B_\wedge)(v,x)=(E,B)(v-|x|,x)$ and therefore they satisfy the equations

$$(1+\widehat{p}\cdot k)\partial_v f_\wedge+\widehat{p}\cdot\nabla_x f_\wedge+(E_\wedge+\widehat{p}\times B_\wedge)\cdot\nabla_p f_\wedge=0, \tag{1.10}$$

$$\partial_v(E_\wedge-k\times B_\wedge)=\nabla\times B_\wedge-j_\wedge, \tag{1.11}$$
$$\partial_v(B_\wedge+k\times E_\wedge)=-\nabla\times E_\wedge, \tag{1.12}$$

$$\partial_v(E_\wedge\cdot k)+\nabla\cdot E_\wedge=\rho_\wedge, \tag{1.13}$$
$$\partial_v(B_\wedge\cdot k)+\nabla\cdot B_\wedge=0, \tag{1.14}$$

where

$$k=\frac{x}{|x|},\quad \rho_\wedge(v,x)=\int f_\wedge\,dp,\quad j_\wedge(v,x)=\int\widehat{p}f_\wedge\,dp. \tag{1.15}$$

Initial data are given at $v=0$ and denoted by

$$f_\wedge^{\mathrm{in}}(x,p)=f_\wedge(0,x,p),\quad E_\wedge^{\mathrm{in}}(x)=E_\wedge(0,x),\quad B_\wedge^{\mathrm{in}}(x)=B_\wedge(0,x).$$

Later we shall discuss the equivalence of the system above with the evolution equations (1.10)-(1.12) and a set of constraint equations on the initial data.

In this paper we are interested in the question of existence and uniqueness of classical solutions in the future (i.e., for $v\in[0,\infty[$) which match the initial data at $v=0$. Obviously, one cannot expect (in general) that a unique solution

is determined by initial data at $v=0$, since the intersection between the initial surface and the domain of dependence of the solutions on a space-time point in the future is not a compact set. However it turns out that the Maxwell equations (1.11)–(1.14) have indeed at most one solution for given data $(E_\wedge^{\mathrm{in}}, B_\wedge^{\mathrm{in}})$ at $v=0$ provided the no-incoming radiation condition is satisfied. This suggests that the solutions we seek to the initial value problem with data on a past light cone should be restricted to the class of solutions isolated from incoming radiation.

This paper is organized as follows. In Section 2 we prove some general properties of smooth solutions to the system (1.10)–(1.15). The results of Section 2 are conditional, as they assume the existence of global classical solutions. In Section 2 we also discuss the relation between the conservation laws satisfied by solutions with data on a past light cone and solutions with data at $t=0$. Note in fact that for solutions with data on a past light cone, the conservation of the total mass and of the total energy are not obvious. In Section 3 we prove global existence and uniqueness of spherically symmetric solutions. This result is obtained by adapting to our case the proof of global existence for the Cauchy problem given in [1, 13]. In spherical symmetry the magnetic field vanishes identically (if decay at infinity is imposed) and the Maxwell equations reduce to the Poisson equation for the electric field. Hence there is neither incoming nor outgoing radiation in spherical symmetry. We will show that, as a consequence of the absence of radiation, spherically symmetric solutions satisfy the conservation laws (1.6), (1.8) and that the total mass-energy equals the mass-energy on the past light cones and on the future light cones.

In a subsequent publication the results of this paper will be extended to the Nordström-Vlasov system (see [4] for a derivation of this model). While it is easy to generalize the formal analysis of Section 2 below to the Nordström-Vlasov system, the proof of global existence and uniqueness of spherically symmetric solutions is different and considerably more involved since the Nordström scalar field equation remains hyperbolic—and so radiation propagates—also in spherical symmetry.

2 The initial value problem with data on a past light cone

An assumption on the initial data which will be made throughout is that

$$0 \leqslant f_\wedge^{\mathrm{in}} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3), \quad E_\wedge^{\mathrm{in}}, B_\wedge^{\mathrm{in}} \in C^2(\mathbb{R}^3)$$

and we define

$$R_0 = \inf\{R : f_\wedge^{\mathrm{in}}(x,p) = 0, |x| \geqslant R, p \in \mathbb{R}^3\}.$$

Hence $f_\wedge^{\mathrm{in}} = 0$, for $|x| \geqslant R_0$. In this section we study several properties of global solutions satisfying the regularity condition

$$f_\wedge \in C^1([0,\infty[\times\mathbb{R}^3\times\mathbb{R}^3), \quad E_\wedge, B_\wedge \in C^1([0,\infty[\times\mathbb{R}^3)$$

and so they are solutions of (1.10)–(1.15) in a classical sense. We also assume that $f_\wedge$ has bounded support in the momentum, precisely

$$\mathcal{P}_\wedge(v)=\sup\{|p|: f_\wedge(s,x,p)\neq 0, 0\leqslant s\leqslant v, x\in\mathbb{R}^3\}<\infty, \quad \forall v\in\mathbb{R}.$$

In particular, all the integrals in the momentum variable in the sequel are understood to be extended over a compact set. We split the analysis in two different subsections.

2.1 The Vlasov equation

We start by pointing out some basic properties of $f_\wedge$. Note the estimate

$$\begin{aligned} 1+\widehat{p}\cdot k &= 1+\frac{p\cdot k}{\sqrt{1+|p|^2}}\geqslant 1-\frac{|p|}{\sqrt{1+|p|^2}} \\ &= \frac{1}{\sqrt{1+|p|^2}(\sqrt{1+|p|^2}+|p|)}\geqslant \frac{1/2}{1+|p|^2}; \end{aligned} \tag{2.1}$$

hence when the support in p of $f_\wedge$ is bounded, the equation (1.10) is equivalent to

$$\partial_v f_\wedge+\frac{p}{p_0}\cdot\nabla_x f_\wedge+\frac{1}{p_0}\left(\sqrt{1+|p|^2}E_\wedge+p\times B_\wedge\right)\cdot\nabla_p f_\wedge=0, \tag{2.2}$$

where p_0 is defined by

$$p_0=\sqrt{1+|p|^2}+p\cdot k>0.$$

The characteristics of the differential operator in the left hand side of (2.2) are the solutions of

$$\dot{x}=\frac{p}{p_0}, \quad \dot{p}=\frac{1}{p_0}\left(\sqrt{1+|p|^2}E_\wedge+p\times B_\wedge\right) \tag{2.3}$$

and we denote by $(X,P)(s,v,x,p)$, or simply $(X,P)(s)$, the characteristic satisfying $(X,P)(v)=(x,p)$. Since the particle density $f_\wedge$ is constant along these curves, we obtain the following representation formula for the solution of the Vlasov equation:

$$f_\wedge(v,x,p)=f_\wedge^{\mathrm{in}}((X,P)(0,v,x,p)). \tag{2.4}$$

In particular $f_\wedge$ remains non-negative for all times and $\|f_\wedge(v)\|_\infty\leqslant\|f_\wedge^{\mathrm{in}}\|_\infty$. In the next lemma we estimate the $x-$support of $f_\wedge$.

Lemma 1 *For all $v\geqslant 0$,*

$$f_\wedge(v,x,p)=0, \quad \textit{for } |x|\geqslant R_0+\frac{1}{2}v.$$

Proof: For all $0\leqslant s\leqslant v$ we have, by the first equation in (2.3),

$$|x|=|X(0)|+\int_0^v\frac{P(\tau)\cdot K(\tau)}{\sqrt{1+|P(\tau)|^2}+P(\tau)\cdot K(\tau)}d\tau,$$

where $K=X/|X|$. Let $[0,v]=\mathcal{I}^-\cup\mathcal{I}^+$, where

$$\mathcal{I}^-=\{\tau\in[0,v]:(P\cdot K)(\tau)\leqslant 0\},\quad \mathcal{I}^+=\{\tau\in[0,v]:(P\cdot K)(\tau)>0\}.$$

Thus, using $\sqrt{1+|p|^2}>p\cdot k$,

$$\begin{aligned}|x| &\leqslant |X(0)|+\int_{\mathcal{I}^+}\frac{P\cdot K}{\sqrt{1+|P|^2}+P\cdot K}d\tau\\ &\leqslant |X(0)|+\frac{1}{2}\mathrm{meas}(\mathcal{I}^+)\leqslant |X(0)|+\frac{1}{2}v.\end{aligned}$$

Since $|X(0)|\leqslant R_0$ in the support of $f_\wedge$, the lemma is proved. □

We shall now derive the conservation laws satisfied by the solutions of (2.2). A straightforward computation reveals that the right hand side of the system (2.3), i.e., the vector

$$F(v,x,p)=\left[\frac{p}{p_0},\frac{1}{p_0}\left(\sqrt{1+|p|^2}E_\wedge+p\times B_\wedge\right)\right],$$

satisfies

$$\left[\nabla_{(x,p)}\cdot F\right](s,X(s),P(s))=-\frac{d}{ds}\log\left(1+\widehat{P}(s)\cdot K(s)\right), \tag{2.5}$$

where $\widehat{P}=P/\sqrt{1+P^2}$. In fact, each side of (2.5) equals, along characteristics,

$$-\frac{1}{(1+\widehat{p}\cdot k)^2}\left[\frac{|\widehat{p}\times k|^2}{|x|}+\frac{1}{\sqrt{1+|p|^2}}\Big(E_\wedge\cdot(k-(\widehat{p}\cdot k)\widehat{p})-(\widehat{p}\times k)\cdot B_\wedge\Big)\right].$$

From (2.5) we deduce

$$\det\left[\frac{\partial(X,P)(s)}{\partial(x,p)}\right]=\frac{1+\widehat{p}\cdot k}{1+\widehat{P}(s)\cdot K(s)}.$$

Hence using (2.4) the next lemma follows.

Lemma 2 *For any measurable function $Q:\mathbb{R}\to\mathbb{R}$,*

$$\int\int Q(f_\wedge)(1+\widehat{p}\cdot k)\,dp\,dx=const.$$

In particular, by choosing $Q(z)=z^q$, $q\geqslant 1$,

$$\|(1+\widehat{p}\cdot k)^{1/q}f_\wedge(v)\|_{L^q(\mathbb{R}^3\times\mathbb{R}^3)}=const. \tag{2.6}$$

The case $q=1$ in (2.6) corresponds to the conservation of the (rest) mass on the past light cones. To be more precise, observe that $\rho_\wedge, j_\wedge$ defined in (1.15) satisfy the equation

$$\partial_v(\rho_\wedge+j_\wedge\cdot k)=-\nabla\cdot j_\wedge. \tag{2.7}$$

The latter can be proved either by using (1.10) or by a simple change of variable in (1.7). We define the mass $N_\wedge(v)$ on the past light cone at time v as

$$N_\wedge(v)=\lim_{r\to\infty}\mathfrak{n}_\wedge(v,r),\quad \mathfrak{n}_\wedge(v,r)=\int_{|x|\leqslant r}(\rho_\wedge+j_\wedge\cdot k)\,dx.$$

Note that the function $\mathfrak{n}_\wedge(v,\cdot)$ is non-decreasing and so the above limits exists. By (2.6), $N_\wedge(v)=N_\wedge(0)$, for all $v\geqslant 0$. The total mass of a solution, given by (1.8), can be rewritten as

$$N(t)=\lim_{r\to\infty}\mathfrak{n}(t,r),\quad \mathfrak{n}(t,r)=\int_{|x|\leqslant r}\rho(t,x)\,dx=\int_{|x|\leqslant r}\rho_\wedge(t+|x|,x)\,dx,\quad t\geqslant 0.$$

In the next lemma we prove a formula which relates the mass functions $N(v)$ and $N_\wedge(v)$.

Lemma 3 *For all* $v\geqslant 0$,

$$\mathfrak{n}(v,r)=\mathfrak{n}_\wedge(v,r)-\int_v^{v+r}\int_{|x|=r}j_\wedge\cdot k(v',x)\,dS_r(x)\,dv'.$$

Proof: Integrating (2.7) between v and $v+|x|$ we get

$$(\rho_\wedge+j_\wedge\cdot k)(v+|x|,x)-(\rho_\wedge+j_\wedge\cdot k)(v,x)=-\int_v^{v+|x|}\nabla\cdot j_\wedge(v',x)\,dv'.$$

Integrating in the region $|x|\leqslant r$ we get

$$\begin{aligned}\int_{|x|\leqslant r}(\rho_\wedge+j_\wedge\cdot k)(v+|x|,x)\,dx\; &= \int_{|x|\leqslant r}(\rho_\wedge+j_\wedge\cdot k)(v,x)\,dx \\ &\quad -\int_{|x|\leqslant r}\int_v^{v+|x|}\nabla\cdot j_\wedge(v',x)\,dv'\,dx.\end{aligned}\tag{2.8}$$

Now we use the identity

$$\nabla\cdot\int_v^{v+|x|}j_\wedge(v',x)\,dv'=(j_\wedge\cdot k)\,(v+|x|,x)+\int_v^{v+|x|}\nabla\cdot j_\wedge(v',x)\,dv'.$$

Substituting into (2.8) and using the Gauss theorem proves the lemma. □

By Lemma 1 all characteristics of the Vlasov equation must cross the surfaces $t=v+|x|=const.$ for all $t\geqslant 0$ in compact sets of x. This means in particular that no mass can be lost at spacelike infinity, which explains why the following lemma holds true.

Lemma 4 *For all* $v\geqslant 0$, $N(v)=N_\wedge(v)=N_\wedge(0)$.

Proof: By Lemma 1 and Lemma 3 we have $\mathfrak{n}(v,r)=\mathfrak{n}_\wedge(v,r)$, for $r>2R_0+v$. The claim follows by letting $r\to\infty$. $\square$

Next we define the mass on the future light cone at time v as

$$N^\vee(v)=\lim_{r\to\infty}\mathfrak{n}^\vee(v,r),\quad \mathfrak{n}^\vee(v,r)=\int_{|x|\leqslant r}(\rho^\vee-j^\vee\cdot k)\,dx,$$

where

$$\rho^\vee(v,x)=\rho_\wedge(v+2|x|,x)=\rho(v+|x|,x),$$
$$j^\vee(v,x)=j_\wedge(v+2|x|,x)=j(v+|x|,x).$$

By a change of variable in (2.7) (or in (1.7)) we have

$$\partial_v(\rho^\vee-j^\vee\cdot k)=-\nabla\cdot j^\vee, \tag{2.9}$$

Integrating (2.7) in time between v and $v+2|x|$ and proceeding as in the proof of Lemma 3 we obtain

$$\mathfrak{n}^\vee(v,r)=\mathfrak{n}_\wedge(v,r)-\int_v^{v+2r}\int_{|x|=r}j_\wedge\cdot k(v',x)\,dS_r(x)\,dv'. \tag{2.10}$$

Moreover, by (2.9), $\mathfrak{n}^\vee$ satisfies the equations

$$\partial_v\mathfrak{n}^\vee=-\int_{|x|=r}j^\vee\cdot k\,dS_r(x),\quad \partial_v\mathfrak{n}^\vee-\partial_r\mathfrak{n}^\vee=-\int_{|x|=r}\rho^\vee\,dS_r(x). \tag{2.11}$$

The evolution of the mass on the future light cones is studied in the following lemma.

Lemma 5 *The function $N^\vee(v)$ is non-increasing, that is,*

$$N^\vee(v_2)\leqslant N^\vee(v_1),\ \forall\, v_1\leqslant v_2.$$

Moreover

$$(i)\quad N^\vee(v_2)=N^\vee(v_1)\ \textit{iff}\ \lim_{r\to\infty}\int_{v_1}^{v_2}\int_{|x|=r}j^\vee\cdot k\,dS_r(x)\,dv=0;$$

$$(ii)\quad N^\vee(v)=N_\wedge(v)\ \textit{iff}\ \lim_{r\to\infty}\int_v^{v+2r}\int_{|x|=r}j_\wedge\cdot k\,dS_r(x)\,dv'=0.$$

Proof: Integrating the second equation in (2.11) along characteristics we have $\mathfrak{n}^\vee(v_2,r-v_2)\leqslant\mathfrak{n}^\vee(v_1,r-v_1)$, for all $v_2\geqslant v_1$ and $r>v_2$. In the limit $r\to\infty$ this implies that $N^\vee$ is non-increasing. The claim (i) follows by integrating in time the first equation in (2.11) on the interval $[v_1,v_2]$ and letting $r\to\infty$, while (ii) follows by (2.10), again in the limit $r\to\infty$. $\square$

We remark that for solutions with data on a past light cone it is not obvious that $N^\vee(v)$ is bounded. Moreover, even if bounded, it needs not to be constant.

If $N^\vee(v_2) < N^\vee(v_1)$, for $0 \leqslant v_1 < v_2$, the difference $N^\vee(v_1) - N^\vee(v_2)$ measures the mass lost at future null infinity in the interval $[v_1, v_2]$ of the advanced time. Finally, even if $N^\vee$ is bounded and constant it is not obvious that it must equal $N_\wedge$, since the limit condition in (ii) of Lemma 5 might not be satisfied.

In the next lemma we show that a sufficient condition for the limits in (i) and (ii) of Lemma 5 to be zero is that the momentum support of $f_\wedge$ is bounded *uniformly* in $v \in \mathbb{R}$, as this condition implies that no particles can reach future null infinity.

Lemma 6 *Assume $\mathcal{P}_\wedge(v) \leqslant D$, for all $v \geqslant 0$ and for some positive constant D. Then, for all $v_1, v_2, v \geqslant 0$,*

$$N^\vee(v_2) = N^\vee(v_1), \quad N^\vee(v) = N_\wedge(v).$$

In particular, by Lemma 4, $N^\vee(v) = N(v) = N_\wedge(v) = N_\wedge(0)$, for all $v \geqslant 0$.

Proof: By the assumption,

$$\sqrt{1+|p|^2} \geqslant \frac{\sqrt{1+D^2}}{D}|p| \geqslant \frac{\sqrt{1+D^2}}{D}(p\cdot k)$$

in the support of $f_\wedge$ and so, as in the proof of Lemma 1,

$$|x| \leqslant |X(0)| + \int_{\mathcal{I}^+} \frac{P(\tau)\cdot K(\tau)}{\sqrt{1+|P(\tau)|^2} + P(\tau)\cdot K(\tau)} d\tau \leqslant R_0 + \frac{D}{D+\sqrt{1+D^2}} v,$$

for all $(x,p) \in \operatorname{supp} f_\wedge(v)$, where $\mathcal{I}^+ = \{\tau \in [0,v] : (P\cdot K)(\tau) > 0\}$. This implies that $f_\wedge(v,x,p) = 0$, for $|x| \geqslant R_0 + av$ and $a \in [0, \frac{1}{2}[$. Hence

$$\int_{v_1}^{v_2} \int_{|x|=r} j^\vee \cdot k \, dS_r(x) \, dv = 0, \text{ for } r > (1-2a)^{-1}(R_0 + av_2),$$

$$\int_{v}^{v+2r} \int_{|x|=r} j_\wedge \cdot k \, dS_r(x) \, dv' = 0, \text{ for } r > (1-2a)^{-1}(R_0 + av).$$

Lemma 4 concludes the proof. □

2.2 The Maxwell equations

We now pass to study some general properties of the electromagnetic field $(E_\wedge, B_\wedge)$. First we show the equivalence of the Vlasov-Maxwell system with a set of evolution equations and a set of constraint equations on the initial data. An important point is that, in the present situation, there are more constraint equations than in the case of the Cauchy problem, since the initial data are given on a characteristic surface. Computing the vector product of (1.11) with the unit vector k, subtracting (1.12) and then using (1.14) we obtain

$$k \times (\nabla \times B_\wedge) - k(\nabla \cdot B_\wedge) + \nabla \times E_\wedge - k \times j_\wedge = 0. \tag{2.12}$$

Moreover, computing the vector product of (1.12) with k, adding (1.11) and then using (1.13) we obtain

$$\nabla \times B_\wedge + k(\nabla \cdot E_\wedge) - k \times (\nabla \times E_\wedge) - \rho_\wedge k - j_\wedge = 0. \tag{2.13}$$

On the other hand, the equation (1.13) follows from (1.11) and (2.13), whereas (1.14) follows from (1.12) and (2.12). Hence the whole set of the Maxwell equations is equivalent to the system composed by (1.11)-(1.12) and (2.12)-(2.13). Clearly, since (2.12)-(2.13) are valid for all times, then they must be imposed at $v=0$ in order to obtain a solution of the initial value problem, i.e., (2.12)-(2.13) are constraint equations on the initial data. However these constraint equations are not totally independent. Let W_1, W_2 denote the left hand side of (2.12) and (2.13), respectively. It is easy to verify the following identities:

$$W_1 = k \times W_2 + k(k \cdot \nabla \times E_\wedge - \nabla \cdot B_\wedge),$$

$$W_2 = -k \times W_1 + k(k \cdot \nabla \times B_\wedge + \nabla \cdot E_\wedge - \rho_\wedge - j_\wedge \cdot k).$$

From this it follows that (2.12)-(2.13) are equivalent to the equations

$$\nabla \cdot B_\wedge = k \cdot \nabla \times E_\wedge, \quad k \cdot \nabla \times B_\wedge + \nabla \cdot E_\wedge = \rho_\wedge + j_\wedge \cdot k, \tag{2.14}$$

together with one of the equations $k \times W_1 = 0$, or $k \times W_2 = 0$, that is

$$k \times [(k \times \nabla \times B_\wedge) + \nabla \times E_\wedge - k \times j_\wedge] = 0 \quad (k \times W_1 = 0), \tag{2.15}$$

or

$$k \times [\nabla \times B_\wedge - k \times \nabla \times E_\wedge - j_\wedge] = 0 \quad (k \times W_2 = 0). \tag{2.16}$$

The following proposition concludes our discussion on the reduction of the Vlasov-Maxwell system to a set of evolution equations and a set of constraint equations on the initial data.

Proposition 1 *The following assertions are equivalent:*

(1) $(f_\wedge, E_\wedge, B_\wedge)$ *is a solution to the initial value problem for (1.10)–(1.14)*

(2) $(f_\wedge, E_\wedge, B_\wedge)$ *is a solution to the initial value problem for (1.10)–(1.14) and the initial data satisfy (2.12)-(2.13)*

(3) $(f_\wedge, E_\wedge, B_\wedge)$ *is a solution to the initial value problem for (1.10)–(1.12) and the initial data satisfy (2.14)-(2.15)*

(4) $(f_\wedge, E_\wedge, B_\wedge)$ *is a solution to the initial value problem for (1.10)–(1.12) and the initial data satisfy (2.14) and (2.16)*

Proof: We already proved that (1)$\Leftrightarrow$(2)$\Rightarrow$(3) and (3)$\Leftrightarrow$(4). Thus it is sufficient to establish (4)$\Rightarrow$(1). It is a simple exercise of vector algebra to show that (2.14) are satisfied for all times provided they are satisfied at time $v=0$ and $E_\wedge, B_\wedge, \rho_\wedge, j_\wedge$ satisfy (1.11), (1.12) and (2.7). The latter holds in virtue of the Vlasov equation (1.10). Moreover, (1.13) follows from (1.11) and the second

equation in (2.14), while (1.14) follows from (1.12) and the first equation in (2.14). Thus $(f_\wedge, E_\wedge, B_\wedge)$ is a solution of (1.10)–(1.14) and since (2.12)-(2.13) are satisfied at $v=0$, then it is also a solution of the initial value problem. $\square$

The no-incoming radiation condition in the coordinates (v,x) reads as in the following

Definition 1 *A global solution of (1.10)–(1.15) is said to satisfy the no-incoming radiation condition (NIRC) if, for all* $v_1, v_2 \geqslant 0$,

$$\lim_{r\to\infty}\int_{v_1}^{v_2}\int_{|x|=r} k\cdot[E_\wedge\times B_\wedge](v,x)\,dS_r(x)\,dv=0.$$

A local solution in the interval $[0,V[$, $V>0$, *satisfies NIRC if the above limit is zero for all* $v_1, v_2 \in [0,V[$.

We impose NIRC only in the future, since our purpose is to study the initial value problem forward in time. Likewise we may introduce the concept of outgoing radiation as in [6].

Definition 2 *The outgoing radiation* $\mathcal{E}_{\rm out}(v_1,v_2)$ *emitted by a (global) solution of the Vlasov-Maxwell system in the interval* $[v_1,v_2]$ *of the advanced time is given by*

$$\mathcal{E}_{\rm out}(v_1,v_2)=\lim_{r\to\infty}\int_{v_1}^{v_2}\int_{|x|=r} k\cdot[E_\wedge\times B_\wedge](v+2r,x)\,dS_r(x)\,dv,$$

provided the limit exists.

The energy identity in the coordinates (v,x) reads

$$\partial_v(e_\wedge+\mathfrak{p}_\wedge\cdot k)=-\nabla\cdot\mathfrak{p}_\wedge, \tag{2.17}$$

where

$$e_\wedge=\int\sqrt{1+|p|^2}f_\wedge\,dp+\frac{1}{2}|E_\wedge|^2+\frac{1}{2}|B_\wedge|^2,\quad \mathfrak{p}_\wedge=\int p\,f_\wedge\,dp+E_\wedge\times B_\wedge.$$

The identity (2.17) can be proved either by a direct calculation using the equations (1.10)–(1.15), or by a simple change of variables in (1.5). Next define

$$\mathfrak{m}_\wedge(v,r)=\int_{|x|\leqslant r}(e_\wedge+\mathfrak{p}_\wedge\cdot k)dx.$$

By (2.17), $\mathfrak{m}_\wedge$ satisfies the equations

$$\partial_v\mathfrak{m}_\wedge=-\int_{|x|=r}\mathfrak{p}_\wedge\cdot k\,dS_r(x),\quad \partial_v\mathfrak{m}_\wedge+\partial_r\mathfrak{m}_\wedge=\int_{|x|=r}e_\wedge dS_r(x). \tag{2.18}$$

We define the energy $M_\wedge(v)$ on the past light cone at time v as

$$M_\wedge(v)=\lim_{r\to\infty}\mathfrak{m}_\wedge(v,r).$$

The function $\mathfrak{m}_\wedge(v,\cdot)$ is non-decreasing and so the above limit exists.

Lemma 7 *$M_\wedge$ is a non-decreasing function:*

$$M_\wedge(v_1) \leqslant M_\wedge(v_2), \quad \forall\, v_1 \leqslant v_2.$$

Moreover if the NIRC is satisfied then $M_\wedge(v)$ is constant for all $v \geqslant 0$.

Proof: For all $v_1 \leqslant v_2$ and $r > v_2 - v_1$ we have, integrating the second equation in (2.18), $\mathfrak{m}_\wedge(v_2, r) \geqslant \mathfrak{m}_\wedge(v_1, r + v_1 - v_2)$ and letting $r \to \infty$ we prove that $M_\wedge$ is non-decreasing. To show that $M_\wedge$ is constant in the absence of incoming radiation, we use that, by the first equation in (2.18),

$$\begin{aligned}\mathfrak{m}_\wedge(v_2, r) - \mathfrak{m}_\wedge(v_1, r) = & -\int_{v_1}^{v_2}\int_{|x|=r}\int p\cdot k f_\wedge\, dp\, dS_r(x)\, dv \\ & -\int_{v_1}^{v_2}\int_{|x|=r} k\cdot[E_\wedge \times B_\wedge]\, dS_r(x)\, dv.\end{aligned}$$

By Lemma 1, the first term in the right hand side vanishes for $r > 2R_0 + v_2$, while the second term tends to zero in the limit $r \to \infty$ by the NIRC. $\square$

From Lemma 7 we obtain the following uniqueness theorem for the Maxwell equations.

Lemma 8 *$(E_\wedge, B_\wedge) \equiv 0$ is the unique C^1 solution of the homogeneous system*

$$\partial_v(E_\wedge - k\times B_\wedge) - \nabla\times B_\wedge = 0, \quad \partial_v(B_\wedge + k\times E_\wedge) + \nabla\times E_\wedge = 0, \qquad (2.19)$$

which satisfies the NIRC and the initial condition $(E_\wedge, B_\wedge)(0, x) = 0$.

Proof: By Lemma 7 we have

$$\begin{aligned}0 &= 2\left[|E_\wedge|^2 + |B_\wedge|^2 + 2(E_\wedge\times B_\wedge)\cdot k\right] \\ &= |E_\wedge\cdot k|^2 + |B_\wedge\cdot k|^2 + |E_\wedge - k\times B_\wedge|^2 + |B_\wedge + k\times E_\wedge|^2.\end{aligned}$$

Hence the solution is a plane wave propagating along the $k-$direction, i.e., the vectors $(E_\wedge, B_\wedge, k)$ form an orthogonal triad. It follows by (2.19) that $\nabla\times E_\wedge = \nabla\times B_\wedge = 0$ and so, by (2.14), $\nabla\cdot E_\wedge = \nabla\cdot B_\wedge = 0$. The claim follows. $\square$

By a standard interpolation argument we obtain

Lemma 9 *If the initial data are chosen such that $M_\wedge(0)$ is bounded and the solution satisfies NIRC, then*

$$\|(\rho_\wedge + j_\wedge\cdot k)(v)\|_{L^{4/3}(\mathbb{R}^3)} \leqslant C M_\wedge(0), \quad \forall\, v \geqslant 0,$$

where C is a positive constant which depends only $\|f_\wedge^{\mathrm{in}}\|_\infty$.

Proof: We write

$$\begin{aligned}\rho_\wedge + j_\wedge\cdot k &= \int_{|p|\leqslant R}(1 + \widehat{p}\cdot k) f_\wedge\, dp + \int_{|p|>R}(1 + \widehat{p}\cdot k) f_\wedge\, dp \\ &\leqslant \frac{8\pi}{3}\|f_\wedge^{\mathrm{in}}\|_\infty R^3 + R^{-1}\int p_0 f_\wedge\, dp \leqslant C\left(\int p_0 f_\wedge\right)^{3/4} \\ &\leqslant C(e_\wedge + \mathfrak{p}_\wedge\cdot k)^{3/4},\end{aligned}$$

where in the second line we choose

$$R=\left(\|f_\wedge^{\mathrm{in}}\|_\infty^{-1}\int p_0 f_\wedge\, dp\right)^{1/4}.$$

The claim follows. $\square$

We shall now briefly discuss the relation between the the total energy and the energy on the past light cones. The total energy (1.6) can be rewritten as

$$M(t)=\lim_{r\to\infty}\mathfrak{m}(t,r),\quad \mathfrak{m}(t,r)=\int_{|x|\leqslant r} e(t,x)\,dx=\int_{|x|\leqslant r} e_\wedge(t+|x|,x)\,dx.$$

By (1.5), $\mathfrak{m}(v,r)$ satisfies the equations

$$\partial_v\mathfrak{m}=-\int_{|x|=r}\mathfrak{p}\cdot k\, dS_r(x), \tag{2.20}$$

$$\partial_v\mathfrak{m}\pm\partial_r\mathfrak{m}=\int_{|x|=r}(\pm e-\mathfrak{p}\cdot k)\, dS_r(x). \tag{2.21}$$

The right hand side of (2.21) is non-negative in the + sign case and non-positive in the − sign case.

Lemma 10 *The total energy is constant, i.e.,*

$$M(v_2)=M(v_1),\quad \forall\, v_1,v_2\geqslant 0.$$

Proof: Integrating (2.21) with the plus sign along the characteristics of $\partial_v+\partial_r$ we obtain $\mathfrak{m}(v_2,v_2+r)\geqslant\mathfrak{m}(v_1,v_1+r)$ for all $v_1\geqslant v_2$, which implies, in the limit $r\to\infty$, $M(v_2)\geqslant M(v_1)$. On the other hand, integrating (2.21) with the minus sign along the characteristics of $\partial_v-\partial_r$ gives $\mathfrak{m}(v_2,r-v_2)\leqslant\mathfrak{m}(v_1,r-v_1)$, for all $v_2\geqslant v_1, r>v_2$ and so, letting $r\to\infty$, $M(v_2)\leqslant M(v_1)$. The claim follows. $\square$

We emphasize that for solutions with data on a past light cone it is not obvious that M is bounded. If it is bounded, then, by Lemma 10, it is conserved. In the latter case, however, the total energy and the energy on the past light cones might be different. To see this consider the equation

$$\mathfrak{m}(v,r)=\mathfrak{m}_\wedge(v,r)-\int_v^{v+r}\int_{|x|=r}\mathfrak{p}_\wedge\cdot k(v',x)\, dS_r(x)\, dv'. \tag{2.22}$$

The latter is obtained by integrating (2.17) in time from v to $v+|x|$ and proceeding as in the proof of Lemma 3. By Lemma 1 and (2.22) we have

$$\mathfrak{m}(v,r)=\mathfrak{m}_\wedge(v,r)-\int_v^{v+r}\int_{|x|=r} k\cdot[E_\wedge^{\mathrm{out}}\times B_\wedge^{\mathrm{out}}](v',x)\, dS_r(x)\, dv', \tag{2.23}$$

for $r>2R_0+v$, where $E_\wedge^{\mathrm{out}},B_\wedge^{\mathrm{out}}$ is the field outside the support of the matter. Hence the answer to the question whether or not $M_\wedge=M$ depends on the decay

of the solutions of (2.19) as $r\to\infty$. As we shall discuss in Section 3, the equality $M_\wedge = M$ holds for spherically symmetric solutions, as in this case the magnetic field vanishes identically. An interesting open question is whether $M_\wedge = M$ holds in general in the absence of incoming radiation.

To conclude this section we study the evolution of the energy on the future light cones. Let

$$e^\vee(v,x) = e_\wedge(v+2r,x) = e(v+r,x),$$
$$\mathfrak{p}^\vee(v,x) = \mathfrak{p}_\wedge(v+2r,x) = \mathfrak{p}(v+r,x),$$

which satisfy the equation

$$\partial_v(e^\vee - \mathfrak{p}^\vee\cdot k) = -\nabla\cdot\mathfrak{p}^\vee.$$

Now let

$$\mathfrak{m}^\vee(v,r) = \int_{|x|\leqslant r}(e^\vee - \mathfrak{p}^\vee\cdot k)\,dx, \quad M^\vee(v) = \lim_{r\to\infty}\mathfrak{m}^\vee(v,r)$$

and note the equations

$$\partial_v\mathfrak{m}^\vee = -\int_{|x|=r}\mathfrak{p}^\vee\cdot k\,dS_r(x), \quad \partial_v\mathfrak{m}^\vee - \partial_r\mathfrak{m}^\vee = -\int_{|x|=r}e^\vee\,dS_r(x), \tag{2.24}$$

$$\mathfrak{m}^\vee(v,r) = \mathfrak{m}_\wedge(v,r) - \int_v^{v+2r}\int_{|x|=r}\mathfrak{p}_\wedge\cdot k(v',x)\,dS_r(x)\,dv'. \tag{2.25}$$

By (2.24)-(2.25) and the same argument as in the proof of Lemma 5 we obtain

Lemma 11 *For all* $v_1\leqslant v_2$,

$$M^\vee(v_2)\leqslant M^\vee(v_1).$$

Moreover

$$(i) \quad M^\vee(v_2) = M^\vee(v_1) \ \textit{iff} \ \lim_{r\to\infty}\int_{v_1}^{v_2}\int_{|x|=r}\mathfrak{p}^\vee\cdot k\,dS_r(x)\,dv = 0;$$

$$(ii) \quad M^\vee(v) = M_\wedge(v) \ \textit{iff} \ \lim_{r\to\infty}\int_v^{v+2r}\int_{|x|=r}\mathfrak{p}_\wedge\cdot k\,dS_r(x)\,dv' = 0.$$

The remarks on $N^\vee$ following the proof of Lemma 5 apply to $M^\vee$ as well. In particular, the difference $M^\vee(v_1) - M^\vee(v_2)$, when it does not vanish, measures the energy dissipated by the system to future null infinity in the interval $[v_1,v_2]$ of the advanced time. By (ii) of Lemma 11, this is the sum of two contributions: an energy lost in form of outgoing radiation by the electromagnetic field (as given in Definition 2) and a kinetic energy carried by the particles, which is given by the limit

$$\lim_{r\to\infty}\int_{v_1}^{v_2}\int_{|x|=r}p\cdot k\,f_\wedge(v+2r,x,p)\,dp\,dS_r(x)\,dv.$$

As in the proof of Lemma 6, the latter term vanishes if the momentum support of $f_\wedge$ is uniformly bounded in $v\in\mathbb{R}$, as in this case no particles can move to future null infinity. Given this interpretation, it is natural to identify $M^\vee$ as the analogue of the Bondi mass in General Relativity, see [2].

3 Spherically symmetric solutions

In spherical symmetry we have $\nabla\times E_\wedge=\nabla\times B_\wedge=0$ and so, by the first equation in (2.14), $\nabla\cdot B_\wedge=0$. Under the additional boundary condition $\lim_{r\to\infty}B_\wedge=0$, this implies that the magnetic field vanishes identically. Moreover, by the second equation in (2.14),

$$\begin{aligned} E_\wedge(v,x) &= \frac{k}{r^2}\int_0^r(\rho_\wedge+j_\wedge\cdot k)(v,r')r'^2\,dr' \\ &= \frac{1}{4\pi}\int\frac{(x-y)}{|x-y|^3}(\rho_\wedge+j_\wedge\cdot k)(v,y)\,dy, \end{aligned} \tag{3.1}$$

the second equality being valid in spherical symmetry. By abuse of notation we use the same symbol to denote a spherically symmetric function in spherical and Cartesian coordinates. The Vlasov equation reduces to

$$\partial_v f_\wedge+\frac{p}{p_0}\cdot\nabla_x f_\wedge+\frac{\sqrt{1+|p|^2}}{p_0}E_\wedge\cdot\nabla_p f_\wedge=0. \tag{3.2}$$

In spherical symmetry the particle density is invariant under proper rotations in phase-space. This allows one to write $f_\wedge=f_\wedge(v,r,w,q)$, where $w=(p\cdot k)\in\mathbb{R}$ and $q=|x\wedge p|^2\geqslant 0$, see [15]. However the Vlasov equation is more conveniently studied in the coordinates (x,p). Note also the conservation of angular momentum: along characteristics,

$$\frac{d}{ds}|x\times p|^2=0. \tag{3.3}$$

In the spherically symmetric case we have the following global existence theorem.

Theorem 1 *Let* $0\leqslant f_\wedge^{\rm in}\in C_c^1(\mathbb{R}^3\times\mathbb{R}^3)$ *be spherically symmetric and satisfy*

$$F=\inf\{|x\times p|^2:(x,p)\in\operatorname{supp}f_\wedge^{\rm in}\}>0; \tag{3.4}$$

there exists a unique, spherically symmetric $f_\wedge\in C^1([0,\infty[\times\mathbb{R}^3\times\mathbb{R}^3)$ *solution of (3.1)-(3.2) such that* $f_\wedge(0,x,p)=f_\wedge^{\rm in}(x,p)$. *Moreover, there exists a constant* $C>0$, *depending only on bounds on the initial datum, such that*

$$\mathcal{P}_\wedge(v)\leqslant C. \tag{3.5}$$

Before giving the proof of Theorem 1, let us observe the following

Corollary 1 *For the solution of Theorem 1,*

$$N^\vee(v)=N(v)=N_\wedge(v)=N_\wedge(0),$$

$$M^\vee(v)=M(v)=M_\wedge(v)=M_\wedge(0).$$

Proof: The equality of the mass functions follows from Lemma 6. Since spherically symmetric solutions are isolated from incoming radiation, then $M_\wedge(v)$ is constant by Lemma 7. Setting $B_\wedge^{\mathrm{out}}=0$ and letting $r\to\infty$ in (2.23), we have $M(v)=M_\wedge(v)$. Hence it remains to show that $M^\vee(v)=M_\wedge(v)$, for all $v\geqslant 0$. For this purpose we use (2.25) with $B_\wedge=0$, that is

$$\mathfrak{m}^\vee(v,r)=\mathfrak{m}_\wedge(v,r)-\int_v^{v+2r}\int_{|x|=r}\int_{|p|\leqslant C} p\cdot k\, f_\wedge(v',x,p)\,dp\,dS_r(x)\,dv';$$

as in the proof of Lemma 6, the integral in the right hand side of this identity vanishes for r large enough and letting $r\to\infty$ concludes the proof. □

The proof of Theorem 1 is formally identical to the proof of global existence for the Cauchy problem with data at time $t=0$ given in [13, Theorem II] (see [15] for the case of two different species of particle). We shall sketch it for the sake of completeness, restricting ourselves to derive the main estimates which lead to the proof. Note however that the assumption (3.4) is not made in [13]. Here the condition (3.4) is used to ensure that the characteristics are C^1 in all the parameters. In fact, due to the presence of the unit vector k, the coefficients of the Vlasov equation are in general discontinuous at $x=0$. But thanks to (3.4) and the conservation of angular momentum, the solution is supported away from the axis $|x|=0$ and so it can be defined in a classical sense in terms of the characteristics. The assumption (3.4) can probably be removed by passing to a weaker solution concept, but we shall not pursue this here.

For the proof of Theorem 1, we denote by C any positive constant which depends only on the initial datum. Moreover we define

$$R_{\min}(v)=\inf\{|x|: f_\wedge(s,x,p)\neq 0,\, 0\leqslant s\leqslant v,\, p\in\mathbb{R}^3\}.$$

By the conservation of angular momentum and (3.4),

$$R_{\min}(v)\geqslant\frac{\sqrt{F}}{\mathcal{P}_\wedge(v)}. \tag{3.6}$$

Hence a bound on the momentum support of $f_\wedge$ implies that the particle density vanishes in a neighbourhood of the axis $|x|=0$. Now observe that, by (2.6) for $q=1$,

$$|E_\wedge(t,x)|\leqslant\frac{N_\wedge}{r^2}.$$

Moreover, the bound $(\rho_\wedge+j_\wedge\cdot k)\leqslant C\mathcal{P}_\wedge(v)^3$, Lemma 9 and Hölder's inequality imply

$$\begin{aligned}|E_\wedge(t,x)| &\leqslant \frac{1}{r^2}\left(\int_0^r(\rho_\wedge+j_\wedge\cdot k)^{4/3}r'^2\right)^{1/3}\left(\int_0^r(\rho_\wedge+j_\wedge\cdot k)^{5/6}r'^2\right)^{2/3}\\ &\leqslant\frac{C}{r^2}\|\rho_\wedge+j_\wedge\cdot k\|_{L^{4/3}}^{4/9}\mathcal{P}_\wedge(v)^{5/3}r^2\leqslant C\mathcal{P}_\wedge(v)^{5/3}.\end{aligned}$$

Next define

$$G(v,r) = -\int_r^\infty \min\left(\frac{N_\wedge}{\lambda^2}, C\mathcal{P}_\wedge(v)^{5/3}\right) d\lambda, \quad v,r \geqslant 0.$$

It follows that $G(v,\cdot) \in C^1$ is increasing and $|E_\wedge(v,x)| \leqslant \partial_r G(v,r)$, for all $v \geqslant 0$. Moreover, and since $\mathcal{P}_\wedge(\cdot)$ is non-decreasing, $\partial_r G(v_1,r) \leqslant \partial_r G(v_2,r)$, for $v_1 \leqslant v_2$. Splitting the integral at $R = \sqrt{N_\wedge}(C\mathcal{P}_\wedge(v)^{5/3})^{-1/2}$ one obtains

$$G(v,0) = -2\sqrt{N_\wedge}\left(C\mathcal{P}_\wedge(v)^{5/3}\right)^{1/2}$$

and therefore, for all $r_1, r_2 \geqslant 0$,

$$|G(v,r_1) - G(v,r_2)| \leqslant |G(v,0)| \leqslant C\mathcal{P}_\wedge(v)^{5/6}.$$

Next we claim that

($\sharp$) *there exists at most one* $v_0 \in [0,\infty[$ *such that* $\frac{d}{ds}|X(s)| = 0$ *and if such* v_0 *exists then* $|X(s)|$ *has an absolute minimum at* $s = v_0$.

This follows because, along characteristics,

$$\frac{d}{ds}|X(s)| = \frac{p\cdot k}{p_0}, \quad \frac{d}{ds}(p\cdot k) = \frac{1}{p_0}\left(\sqrt{1+|p|^2}\,E_\wedge\cdot k + \frac{|p\times k|^2}{|x|}\right) > 0.$$

Since $p\cdot k$ is increasing and $\frac{d}{ds}|X(s)| > 0$ (resp. <0) for $p\cdot k > 0$ (resp. <0), the claim ($\sharp$) is proved. Now observe that, along characteristics,

$$\frac{d}{ds}\sqrt{1+p^2} = \frac{p\cdot k}{p_0}|E_\wedge|.$$

Hence, for all $v \geqslant 0$ and $s \in [0,v]$ we have, denoting $\mathcal{I}^+ = \{\tau \in [0,s] : \frac{d}{d\tau}|X(\tau)| \geqslant 0\}$,

$$\begin{aligned}
\sqrt{1+P(s)^2} - \sqrt{1+P(0)^2} &= \int_0^s |E_\wedge(\tau, X(\tau))| \frac{P(\tau)\cdot K(\tau)}{P_0(\tau)} d\tau \\
&= \int_0^s |E_\wedge(\tau, X(\tau))| \frac{d}{d\tau}|X(\tau)| d\tau \\
&\leqslant \int_{\mathcal{I}^+} \partial_r G(\tau, |X(\tau)|) \frac{d}{d\tau}|X(\tau)| d\tau \\
&\leqslant \int_{\mathcal{I}^+} \partial_r G(s, |X(\tau)|) \frac{d}{d\tau}|X(\tau)| d\tau \\
&= \int_{\mathcal{I}^+} \frac{d}{d\tau}[G(s,|X(\tau)|)]\, d\tau.
\end{aligned}$$

By virtue of ($\sharp$), either $\mathcal{I}^+ = [s_1, s]$, for some $0 < s_1 < s$, or $\mathcal{I}^+ = [0,s]$, or $\mathcal{I}^+$ is empty. In the first case we obtain

$$\begin{aligned}
\sqrt{1+P(s)^2} &\leqslant \sqrt{1+P(0)^2} + G(s,|X(s)|) - G(s,|X(s_1)|) \\
&\leqslant \sqrt{1+P(0)^2} + C\mathcal{P}_\wedge(v)^{5/6}
\end{aligned}$$

and since this is true for all $0 \leqslant s \leqslant v$, then $\mathcal{P}_\wedge(v) \leqslant C(1+\mathcal{P}_\wedge(v)^{5/6})$, which implies $\mathcal{P}_\wedge(v) \leqslant C$. The other two cases lead to the same inequality. The bound on the momentum support of $f_\wedge$ implies, by (3.6), that the particle density is supported away from the axis $|x|=0$. This allows one to define $f_\wedge$ in terms of the characteristics and derive L^∞ estimates for its derivatives. A standard iteration scheme completes the proof of the theorem.

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